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LETTER TO THE EDITOR

A note on angular momentum and sums over classical paths

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Abstract. The spherical top propagator and spectral operator kernel for all spins are evaluated in terms of new rotational coordinates that are related to the Euler angles in a simple way. Both operator kernels are expressible exactly as sums over classical paths. The angular momentum wavefunctions are single-valued functions of the new rotational coordinates for all values of j.

A path integral for spin, based on Feynman's formulation of quantum mechanics, has been given by Schulman (1968). The classical model which is quantized by path integration is the spherically symmetric top described by Euler angles. The quantum-mechanical propagator which results from this procedure propagates all spins and, rather surprisingly, contains only classical-path terms. More recently Norcliffe (1972) has evaluated the spectral operator kernel for the spherical top and it too is expressible exactly as a sum over classical paths. These two treatments not only relate the classical and quantum-mechanical theories of angular momentum and spin in a direct way, but they also provide a description of spin in terms of rotational coordinates without involving double-valued wavefunctions explicitly.

If one expresses the J^2 operator in terms of Euler angles, then in this representation the angular momentum wavefunctions corresponding to integer values of j are single valued. The wavefunctions corresponding to half-integer j are double valued. This double valuedness is regarded as unphysical and is one difficulty of treating angular momentum in terms of rotational coordinates. In path-integral theories, where one is dealing directly with the paths, the role of boundary conditions is played by phase factors that are associated with each path contribution. For the spherical top the phases associated with the paths that give rise to half-integer values of j are different to those for integer values (see Schulman 1968, equation (4.19) and Norcliffe 1972 equations (18) and (22)). One might argue that the phases associated with half-integer values of j are also unphysical because they too would lead to double-valued wavefunctions. In this letter we consider the spherical top in a new representation in which the propagator and the spectral operator kernel for *all* spins can be expressed once again as sums over classical paths, but where the respective path contributions are each added with the *same* overall phase. The corresponding angular momentum wavefunctions are also single valued for all values of j in this representation.

Any rotation of a rigid body may be specified by three Euler angles ($\alpha\beta\gamma$) or by four Euler parameters ($\lambda\mu\nu\rho$) (Synge 1960). The Euler parameters satisfy

$$\lambda^2+\mu^2+\nu^2+\rho^2 = 1 \tag{1}$$

and are related to the Euler angles as follows

$$\begin{aligned} \lambda &= \sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma-\alpha}{2}\right), & \nu &= \cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha+\gamma}{2}\right) \\ \mu &= \sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma-\alpha}{2}\right), & \rho &= \cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right). \end{aligned} \tag{2}$$

Any point on the surface of a unit hypersphere in four dimensions thus specifies a rotation uniquely, and instead of the Euler angles we choose as rotational coordinates the angles xyz, $0 \leqslant x, y \leqslant 2\pi$, $0 \leqslant z \leqslant \frac{1}{2}\pi$, for which $x = (\gamma+\alpha)/2$, $y = (\gamma-\alpha)/2$, $z = \beta/2$ and

$$\begin{aligned} \lambda &= \sin z \sin y, & \nu &= \cos z \sin x \\ \mu &= \sin z \cos y, & \rho &= \cos z \cos x. \end{aligned} \tag{3}$$

Such a parametrization of the unit hypersphere has been used by Kuznetsov (1967) in connection with the O(4) symmetry of the hydrogen atom.

In terms of the angles xyz the spherical-top lagrangian function is given by

$$L = 2I(\dot{x}^2\cos^2 z + \dot{y}^2\sin^2 z + \dot{z}^2) \tag{4}$$

where I is the moment of inertia of the top about any axis through its centre of mass. The generalized momenta are given by $p_x = \partial L/\partial \dot{x}$ etc, and the corresponding hamiltonian function is

$$H = \frac{1}{8I}\left(\frac{p_x^2}{\cos^2 z} + \frac{p_y^2}{\sin^2 z} + p_z^2\right). \tag{5}$$

A least-action principle indicates that the trajectories on the hypersphere are geodesics so that as the top rotates in real space, the point $\boldsymbol{R} = (\lambda\mu\nu\rho)$ traces out a great circle on the hypersphere. Between two points on the hypersphere the following action functions take the values:

$$\left.\begin{aligned} \int_{t_0}^{t} L\,dt &= 2I(\omega+2\pi c)^2 = S_c(\boldsymbol{R},t;\boldsymbol{R}_0,t_0) \\ \int_{\boldsymbol{R}_0}^{\boldsymbol{R}} (p_x dx + p_y dy + p_z dz) &= 2k|\omega+2\pi c| = S_{ck}(\boldsymbol{R},\boldsymbol{R}_0) \end{aligned}\right\} c = 0, \pm 1, \ldots \tag{6}$$

where ω is the angle between the two four-vectors $\boldsymbol{R}_0$ and $\boldsymbol{R}$, and k is the magnitude of the classical angular momentum given by $H = k^2/2I$. The actions are multivalued because there is more than one classical trajectory joining two points on the hypersphere.

To obtain the propagator one could path integrate the top knowing the lagrangian function (4). We obtain the propagator here by summing over its stationary states. First we consider the quantum mechanics of the top in terms of the new rotational coordinates. The hamiltonian operator, $H = J^2/2I$, in terms of xyz becomes

$$H = -\frac{\hbar^2}{8I}\left(\frac{1}{\sin z \cos z}\frac{\partial}{\partial z}\sin z \cos z\frac{\partial}{\partial z} + \frac{1}{\cos^2 z}\frac{\partial^2}{\partial x^2} + \frac{1}{\sin^2 z}\frac{\partial^2}{\partial y^2}\right) = \frac{-\boldsymbol{\Delta}\hbar^2}{8I} \tag{7}$$

where $\boldsymbol{\Delta}$ is the Laplace–Beltrami operator on the hypersphere (see for example Pajas and Raczka 1968). The eigenvalues of $\boldsymbol{\Delta}$, corresponding to single-valued eigenfunctions, are of the form $4j(j+1), j = 0, \frac{1}{2}, 1, \ldots$. The eigenfunctions are just the hyperspherical harmonics and from the addition theorem which they satisfy (eg Bander and Itzykson 1966) it follows that the kernel of the projection operator P_j onto the level j is

$$\langle \boldsymbol{R}|P_j|\boldsymbol{R}_0\rangle = \frac{2j+1}{2\pi^2}\,\frac{\sin(2j+1)\omega}{\sin\omega}. \tag{8}$$

The propagator of the spherical top for all spins is thus

$$\left\langle \boldsymbol{R}\left|\exp\frac{-\mathrm{i}}{\hbar}H(t-t_0)\right|\boldsymbol{R}_0\right\rangle = \sum_{j=0,\frac{1}{2},1,\ldots} \exp\frac{-\mathrm{i}}{\hbar}\,\frac{j(j+1)\hbar^2}{2I}(t-t_0)\,\langle \boldsymbol{R}|P_j|\boldsymbol{R}_0\rangle$$
$$= \sum_j \frac{2j+1}{2\pi^2}\,\frac{\sin(2j+1)\omega}{\sin\omega}\exp\frac{-\mathrm{i}\hbar}{2I}j(j+1)(t-t_0). \tag{9}$$

After some analysis, and use of the Poisson summation formula (cf Schulman 1968 equation (3.7)), the expression for the propagator finally emerges as a sum over classical paths

$$\left\langle \boldsymbol{R}\left|\exp\frac{-\mathrm{i}}{\hbar}H(t-t_0)\right|\boldsymbol{R}_0\right\rangle$$
$$= \sum_{c=-\infty}^{\infty} \frac{4(\omega+2\pi c)\exp\{\mathrm{i}\hbar(t-t_0)/8I\}}{\sin\omega}\left(\frac{I}{2\pi\mathrm{i}\hbar(t-t_0)}\right)^{\frac{3}{2}}\exp\left(\frac{\mathrm{i}}{\hbar}S_c(\boldsymbol{R},t;\boldsymbol{R}_0,t)\right). \tag{10}$$

All the classical path contributions are added together with the same overall phase. Similarly if we evaluate the spectral operator kernel we see that

$$\langle \boldsymbol{R}|\delta(E-H)|\boldsymbol{R}_0\rangle = \sum_{c=-\infty}^{\infty} \frac{I}{\pi^2\sin\omega}\,\frac{|c|}{c}\sin\frac{S_{ck}(\boldsymbol{R},\boldsymbol{R}_0)}{\hbar} \tag{11}$$

is also a classical path sum where all the terms contribute with the same overall phase factor $|c|/c$ for each path.

The reason why only *one* overall phase factor in the path sums (10) and (11) is needed for each operator stems from the fact that the wavefunctions (hyperspherical harmonics) for all values of j are single-valued functions over the hypersphere. Indeed, the theory of total angular momentum can be formulated without the need for spinors if the rotational coordinates xyz are used.

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